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Use of the delta method to evaluate the precision of assessments that fix parameter values

Alec D. MacCall

National Marine Fisheries Service, Southwest Fisheries Science Center, Fisheries Ecology Division, 110 Shaffer Road, Santa Cruz, CA 95060, USA

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ABSTRACT

Many stock assessments fix values for influential, but poorly known, parameters such as the natural mortality rate or the steepness of the stock-recruitment relationship, which leads to published estimates of uncertainty being underestimates. The delta method, in which the partial derivatives of the model outputs with respect to all of the parameters of the model can be easily obtained numerically using likelihood profiling, can be used to provide quick, but approximate, estimates of precision for model outputs when some key parameters are fixed.

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1. Introduction

There has been a progressive improvement in the ability to estimate the precision of management quantities reported in fish stock assessments given the steadily increasing computing power as well as ongoing software development. Moreover, in the United States, the need to estimate precision has recently become urgent due to passage of a new administrative requirement to account for "scientific uncertainty" in the setting of annual catch limits (Department of Commerce, 2009). However, some critically important parameters are conventionally fixed rather than estimated within assessments. Ironically, the values of these parameters are fixed often because they are poorly known or are difficult to estimate from available data, but their treatment as fixed has the statistical implication that they are known without error! About half of the assessments of rockfishes (*Sebastes* spp.) off the US west coast fix the value for stock-recruitment steepness (h) based on a meta-analysis by Dorn (2002) and subsequent unpublished updates. Most of these assessments also fix the value of the natural mortality rate (M), and nearly all fix the value of σ_R, the precision statistic describing the variability of recruitment anomalies about the fitted stock-recruitment relationship (SRR). The uncertainty associated with these fixed parameters is rarely addressed other than by a tabulated sensitivity analysis, and does not appear in the reported estimates of precision. However, the variance of, for example, reported terminal year biomass estimates among a sequence of stock assessments for the same stock is known to be much larger than the reported within-assessment estimation variance for any one assessment. For example, Ralston et al. (2011) found that the among-assessment coefficient of variation (CV) in biomass estimates was 37% for 17 data-rich assessments of U.S. west coast groundfish and coastal pelagic fishes, but the within-assessment "reported" CV was only 18%.

Modern stock assessment packages often provide variance estimates that quantify uncertainty, conditionally on any fixed parameters. Resampling (bootstrap) analysis is used in the case of virtual population analysis (VPA), while inversion of the Hessian matrix is used with maximum likelihood models such as Stock Synthesis. Sampling of parameter vectors from Bayesian posteriors using the Markov chain Monte Carlo (MCMC) algorithm is now popular, and is becoming a preferred approach (Magnusson et al., in press). However, these approaches require that all influential parameters be estimated, and can produce severely negatively biased estimates of variance when uncertain parameters are fixed.

The delta method for estimating the variance of functions of parameters was elaborated by Cramér (1946), and was promoted in operations research by Koopman (1946) and Morse and Kimball (1951), this being why Hilborn and Mangel (1997) refer to the delta method as the method of "navy math." The delta method was popularized in animal ecology by Seber (1973). This method can be argued to be a logical extension of the sensitivity analyses that are often included in modern stock assessments. This paper demonstrates that the delta method is a practical approach for obtaining approximate estimates of variance for stock assessments that fix parameters.

A stock assessment (or more specifically, any of its outputs) can be viewed as an elaborate function to which the delta method can be applied. Applications of the delta method to VPA-based stock assessments were reported by Sampson (1987), Prager and MacCall (1988), Pelletier (1990), and recently by Hillary (in press). Maunder et al. (2006) used the delta method as a parametric alternative

E-mail address: Alec.MacCall@noaa.gov

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to stochastic projections of the uncertainty in future stock status. However, as a case in point, that investigation fixed M and did not use the delta method to explore uncertainty in M as a source of imprecision in the projections. Magnusson et al. (in press) found that the performance of the delta method was fairly comparable to that of MCMC, but their investigation was restricted to parameters that were estimated.

2. The delta method

2.1. Theory

Seber (1973) gives a relatively transparent form of the delta method for approximating the variance of a function $g(\bar{x})$:

$$V[g(\vec{x})] \approx \sum_{i=1}^{n} V[x_i] \left(\frac{\partial g(\vec{x})}{\partial x_i}\right)^2 + 2\sum_{i<j}\sum cov[x_i, x_j] \times \left(\frac{\partial g(\vec{x})}{\partial x_i}\right)\left(\frac{\partial g(\vec{x})}{\partial x_j}\right) \quad (1)$$

where $\vec{x}$ denotes a vector of parameters with elements x_i (i = 1, 2, 3, ...), and $g(\vec{x})$ is a function of those random variables. The delta method is based on a Taylor series expansion of the function g, and is evaluated at the means of the individual elements. In Eq. (1), bias and higher order (quadratic) terms in the Taylor expansion are ignored. In the applications presently being considered, the parameters $\vec{x}$ are those parameters that are fixed, typically M, σ_R, and an SRR shape parameter such as h (defined by Mace and Doonan, 1988), though other fixed parameters may also be considered; importantly, in the context of the delta method, all of the parameters estimated by the model are treated as "nuisance parameters" that may need to be re-estimated to maintain the property of conditional maximum likelihood given alternative values of the fixed parameters. The function g represents an estimated quantity produced by the assessment, such as current spawning stock biomass ($SSB_{current}$), management reference points such as the unfished biomass ($SSB_{unfished}$) or the biomass corresponding to maximum sustainable yield (B_{msy}), and composite status indicators such as current relative stock size ($SSB_{current}/SSB_{unfished}$). The original fixed-parameter model is denoted as $g(\cdot)$ and the conditional variance estimate obtained from the model before application of the delta method as $V[g(\cdot)]$. This conditional variance is treated as independent of the fixed parameters.

Eq. (1) is a sum of estimated variance components $\widehat{V}[g(x_i)]$ in the simple, but common, case where the fixed parameters are $\vec{x} = (M, h, \sigma_R)$, and they are assumed to be independent, i.e., Cov(M, h) = 0. As shown in Eq. (1), each individual variance component is estimated as the product of the variance of the fixed parameter and the square of the partial derivative of the estimated assessment quantity $g(x_i)$ with regard to that parameter. Accordingly, in the simple case of independence among the fixed parameters, the delta method estimate of the variance of an estimated quantity g becomes:

$$V[g(\vec{x})] \approx \widehat{V}[g(\cdot)] + \widehat{V}[g(M)] + \widehat{V}[g(h)] + \widehat{V}[g(\sigma_R)] \quad (2)$$

Eq. (2) allows the relative contribution from each source of variance to the overall variance to be calculated because the total variance is the sum of individual components. The contribution of covariances among fixed parameters can be negative, which complicates nominal partitioning of variance components.

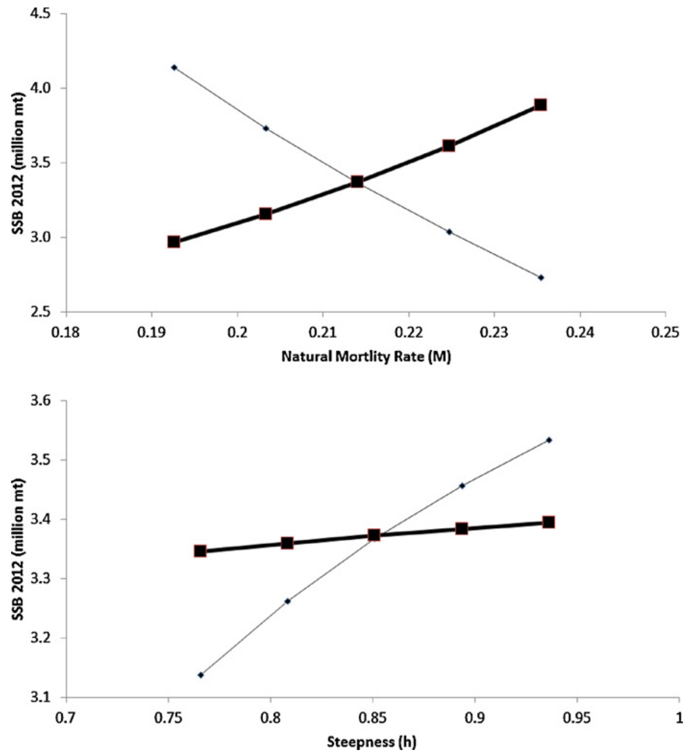

Fig. 1. Examples of numerical calculation of partial derivatives. Thick line is likelihood profile, thin line is finite difference. Upper: Profile ∂(SSB)/∂M = 21.5E6 yr-tons, Finite difference ∂(SSB)/∂M = −32.9E6 yr-tons. Lower: Profile ∂(SSB)/∂h = 0.286E6 tons, Finite difference ∂(SSB)/∂h = 2.33E6 tons.

2.2. Calculation

Appendix A describes a spreadsheet for performing the calculations. The two main issues are calculation of partial derivatives and specification of prior variances and covariances.

2.2.1. Derivatives

Partial derivatives are easily calculated using numerical methods. The assessment model is re-run (using identical convergence criteria) at slightly higher and lower values for each fixed parameter, and partial derivatives are calculated numerically. Using perturbations of equal distance Δ above and below the fixed parameter value, the approximate partial derivative is calculated numerically as a slope (Abramowitz and Stegun, 1972) by:

$$\frac{\partial g(\vec{x})}{\partial x_i} \approx \frac{g(x_i + \Delta) - g(x_i - \Delta)}{2\Delta} \quad (3)$$

where other parameters $x_{j \neq i}$ are held at the fixed values used in the assessment. In practice, it is advisable to compare results from alternative values of Δ to ensure that the estimated derivative is a good estimate of the slope of the function in the vicinity of the pre-specified value for the fixed parameter (Fig. 1).

There are two possible ways to calculate this partial derivative in SS. A strict "finite difference" approach would hold all nuisance parameters fixed while changing only the single parameter x_i. The "profile likelihood" allows nuisance parameters to be re-estimated to achieve a conditional maximum posterior density given the alternative values of $x_i \pm \Delta$ (for technical aspects of these approaches, see Cox (1975), and Murphy and van der Vaart (2000)). In either case, parameters $x_{j \neq i}$ are kept at their fixed values. The partial derivatives obtained using these two methods need not be similar (Fig. 1), and could give quite different results when used in the delta method. I argue that the likelihood profile approach is correct for use in the delta method. From a fundamental

Table 1
Precision of Pacific whiting stock assessment. Biomasses are in million tons.

Quantity	femSSB_{2011}	relSPR_{2010}	$\text{SSB}_{2011}/\text{SSB}_{\text{unfished}}$
"SS free" MPD estimate[a]	1.69	0.695	0.89
CV SS free[a]	0.428	0.244	0.382
Lower (25 percentile)	1.20	0.58	0.66
Upper (75 percentile)	2.17	0.81	1.12
50% interval width	0.97	0.23	0.46
MCMC mean estimate[a]	1.87	0.64	0.91
Lower (25 percentile)	1.33	0.51	0.68
Upper (75 percentile)	2.65	0.77	1.23
50% interval width	1.32	0.26	0.55
Delta method with $\sigma_{\ln(M)} = 0.1$, using likelihood profile			
"SS fixed" MPD estimate	1.69	0.695	0.89
CV base (from SS fixed)	0.411	0.209	0.372
Variance portion from base	89%	68%	85%
Variance portion from M	10%	31%	5%
Variance portion from h	0%	0%	1%
Variance portion from σ_R	1%	1%	9%
CV delta method[b]	0.435	0.253	0.403
50% confidence interval			
Lower (25 percentile)	1.19	0.58	0.65
Upper (75 percentile)	2.18	0.81	1.13
50% interval width	0.99	0.23	0.48
Delta method with $\sigma_{\ln(M)} = 0.1$, using finite difference (*incorrect*)			
CV delta method	0.528	0.457	0.577

[a] "SS free" and MCMC use a fixed value of σ_R (see Stewart et al., 2013).
[b] CVs are 0.433, 0.252, and 0.384 if σ_R is fixed at 1.3 as in the "SS free" model.

viewpoint, maximum likelihood parameter estimates are explicitly conditional on the data given to the model. The likelihood profile approach preserves this property, while the finite difference approach loses all connection to the underlying data. From a practical viewpoint, the likelihood profile allows the delta method to produce very similar estimates to the corresponding asymptotic estimates if the parameter is estimated, while the partial derivatives from finite differencing do not (Table 1).

2.2.2. Variances

Variances for the parameters in Eq. (1) can be difficult to specify, and their coefficients of variance may have to be based on "educated guesses". Prior probability distributions for model parameters may have been developed in some cases, such as from meta-analysis. Meta-analyses for h exist both on a regional scale (e.g., Dorn, 2002; Forrest et al., 2010) and on a worldwide scale (Myers et al., 1999, 2002). Approximate variances for h can be derived from re-analysis of information on distribution percentiles (Dorn, 2002) or from confidence limits (Myers et al., 1999). If it is not possible to derive a variance for h from such sources, Dorn's (2002) results suggest a tentative rule-of-thumb that the approximate value of the standard error (σ_h) for Beverton–Holt SRR steepness is half the distance to the nearest boundary (the bounds being $0.2 \le h \le 1$). For example if h is fixed at 0.6, $\sigma_h \approx 0.2$, and if h is fixed at 0.8, $\sigma_h \approx 0.1$. The author is unaware of meta-analyses that provide a useful prior probability distribution for σ_R, but tentative values can be based on ad-hoc comparisons with assessments for similar species.

Two well-established empirical relationships for estimating M are those of Pauly (1980) and Hoenig (1983). Pauly gives a $\sigma_{\ln(M)}$ of 0.56 for estimated $\ln(M)$ based on growth and temperature (note: this value has been converted from the original value which was given in units of $\log_{10}$). Hoenig's relationship between M and maximum observed age did not include an estimate of precision, but re-analysis of his original data for fish (Hoenig, 1982) reproduces the original regression parameters, and gives a $\sigma_{\ln(M)}$ of 0.50. Another recent growth-based method for estimating M developed by Gislason et al. (2010) gives a $\sigma_{\ln(M)}$ of 0.72, which is less precise

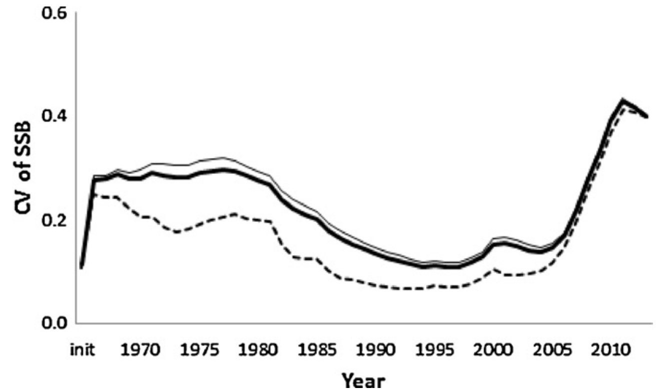

Fig. 2. Time-trajectories of estimated coefficients of variation (CV) for the spawning stock biomass (SSB) of Pacific whiting. Heavy solid line is "SS free" (M and h are estimated); lower dashed line is "SS fixed" (M and h are fixed), light solid line is the delta method applied to "SS fixed" with $\sigma_{\ln(M)} = 0.1$.

than the relationships developed by Pauly or Hoenig. The CV of M can easily be obtained from $\sigma_{\ln(M)}$: for a lognormal distribution, Johnson et al. (1994) give the coefficient of variation of a lognormal variate as $\text{CV} = (\exp(\sigma^2) - 1)^{1/2}$, so if $\sigma(\ln M) = 0.5$, then $\text{CV}(M) = 0.53$.

The covariances among M, h and σ_R are especially difficult to obtain, but there is some support for ignoring covariances. Myers et al. (2002) examined 246 assessed fish stocks, and concluded that h is negatively correlated with M only for short-lived species, with reproductive longevity up to 5 yrs. A meta-analysis by Shertzer and Conn (2012) also failed to find a relationship between h and M. This suggests that the term $\text{cov}[M, h]$ in Eq. (1) can be ignored except for short-lived fishes. To the author's knowledge, there is no evidence that σ_R is strongly correlated with either M or h.

3. Examples

Two examples are explored. The Stock Synthesis assessment for Pacific whiting (*Merluccius productus*) also allows comparison with MCMC results, and is taken from Stewart et al. (2013). The delta method calculations for Pacific whiting follow the spreadsheet outlined in Appendix A. A VPA assessment for Georges Bank haddock (*Melanogrammus aeglefinus*) is taken from Brooks et al. (2008); additional model runs were provided by E. Brooks (NMFS-NEFSC, Pers. Comm.).

3.1. Pacific whiting

Stewart et al. (2013) quantify the uncertainty of model outputs from an SS model using asymptotic standard errors as well as MCMC sampling of this model cast in a Bayesian framework. Their SS model ("SS free") estimated M and h (both subject to priors), but fixed $\sigma_R = 1.3$. I developed an alternative model ("SS fixed") where M and h were fixed at their maximum posterior density estimates ($0.214\,\text{yr}^{-1}$ and 0.851), emulating a common practice in more data-limited US west coast groundfish assessments. The delta method applied with the relatively precise prior of $\sigma_{\ln(M)} = 0.1$ adopted by Stewart et al. (2013) to the "fixed" model leads to CVs that are quite close to the asymptotic CVs from the "free" model, clearly demonstrating the performance of the approximation (Fig. 2). In contrast, and as expected, the asymptotic CVs from the "fixed" model were tighter than the delta method CVs (Table 1 and Fig. 1). The delta method estimates for Pacific whiting also address imprecision in σ_R, a parameter that was fixed by Stewart et al. (2013). The delta method results shown in Table 1 indicate that the additional variance due to uncertainty in σ_R is negligible for estimated female biomass in 2011 (femSSB_{2011}) and the relative spawning potential

Table 2
Precision of Georges Bank haddock stock assessment. Biomasses are in 10^5 tons.

Quantity (g)	SSB_{2007}	F_{2007}	SSB_{msy}	SSB_{2007}/SSB_{msy}
CV VPA bootstrap	0.199	0.165	0.26	0.199
Base variance $V[g(\cdot)]$	0.394	0.00144	0.173	0.156
Change in function values used in the calculation of partial derivatives				
$g(M=0.18\,yr^{-1})$	2.98	0.239	1.33	2.24
$g(M=0.20\,yr^{-1})$ (base)	3.16	0.230	1.59	1.99
$g(M=0.22\,yr^{-1})$	3.37	0.219	1.59	2.11
Partial derivative	9.75	−0.500	6.50	−3.25
Delta method with $\sigma_{\ln(M)}=0.5$ (CV = 0.53)				
Variance from M	1.068	0.00281	0.475	0.119
Total variance	1.462	0.00425	0.648	0.275
Portion from base	27%	34%	27%	57%
Portion from M	73%	66%	73%	43%
CV Delta method	0.384	0.287	0.518	0.264

ratio in 2010 ($relSPR_{2010}$), but has a small (9% of total estimated variance) impact on the CV of the estimated relative biomass in 2011 ($SSB_{2011}/SSB_{unfished}$).

The delta method, in common with the asymptotic precision estimates from SS (provided in the SS3.std, SS3.cor and covar.sso files), is based on a normal approximation in the immediate vicinity of the point estimate to the sampling distribution for a model output. These analytic methods provide no information about the shape or tails of the distributions. For this reason it is more useful to describe a tentative 50% confidence interval or inter-quartile range rather than to extrapolate to a conventional 95% confidence interval as the latter makes strong assumptions about the shape of the distribution. The MCMC credibility intervals from Stewart et al. (2013) provide a more thorough description of uncertainty, including the shape of the distribution and shifts in central tendency (mode or mean). However, except for shifts in central tendency, the delta method leads to inter-quartile ranges that are similar to both the asymptotic estimates from Stock Synthesis with estimated parameters and those from MCMC sampling (Table 1).

3.2. Georges Bank haddock

This example (Table 2) is drawn from the "GARM3" collection of groundfish assessments (Brooks et al., 2008), but it must be noted that the assessment was subsequently revised (L. Brooks, NEFSC, Pers. Comm.). The results presented here are therefore only illustrative. The assessment is based on aged landings and extensive fishery-independent trawl surveys, and utilizes VPA with a fixed natural mortality rate of $M = 0.2\,yr^{-1}$. The reported results appear to be precise, based on bootstrap resampling of age compositions, but are conditional on the fixed value of M. Management reference points are estimated externally to the stock assessment. The target fishing mortality rate (F_{msy}) is a proxy value based on a spawning potential ratio (SPR) of 40% and recent selectivity patterns, and is reported implicitly as being exact. The target biomass (SSB_{msy}) is estimated by conducting Monte Carlo forward projections of the recent recruitments from the assessment model to equilibrium under the proxy F_{msy} fishing mortality rate, using the empirical distribution of recruitment to spawning stock biomass ratios. Brooks et al. (2008) report a mean projected SSB_{msy} of 158,000 tons, with a 90% confidence interval (CI) of 96,000–230,000 tons. Details of the distribution are not reported, but the 90% CI is nearly symmetrical at ±42% of the mean, and translates to an approximate CV of 0.26 assuming normality.

The delta method was used to estimate the imprecision associated with the fixed natural mortality rate, using the Hoenig-based prior $\sigma_{\ln(M)} = 0.5$ or CV(M) = 0.53. Although a tighter CV for M could be considered, the entrenched practice of assuming the same value of $M = 0.2\,yr^{-1}$ for the majority of New England stocks suggests that the full range of ancillary evidence is not being considered and therefore the CV is not as small as it could otherwise be. VPA is not a likelihood-based approach, and does not support internal estimation of M, precluding improvement in posterior precision.

Table 2 includes some details of the implementation of the delta method. The bootstrap precision reported in the assessment is taken as the base variance $V[g(\cdot)]$. M is the only fixed parameter. The delta method CVs for haddock spawning stock biomass (SSB_{2007}) and current fishing mortality rate (F_{2007}) are about double those reported from the bootstrap analysis, and the majority of the variance is associated with M. The uncertainty about target biomass is strongly influenced by uncertainty in M, with a delta method CV of 0.518. However, current stock abundance relative to the target abundance has a much tighter delta method CV of 0.265 because of cancellation of similar effects of M on estimated biomass in the numerator and denominator of the ratio (i.e., covariance).

Table 2 also demonstrates a problem that arises frequently in numerical calculation of partial derivatives, especially in the absence of rigorous criteria for model convergence. The SSB_{2007} and F_{2007} have a fairly linear trend over the range of $M \pm 0.02$, but SSB_{msy} has a much steeper slope (local partial derivative) on the lower side of M (which could possibly be an artifact of the projection methodology used for haddock), and SSB_{2007}/SSB_{msy}, has opposite signs of slopes above and below $M = 0.2\,yr^{-1}$! In this case, the delta method attempts to project the effect of imprecision in M into regions where the partial derivative at $M = 0.2\,yr^{-1}$ does not accurately describe the nearby response. A Monte Carlo exploration of the effect of imprecise M would be a better approach for evaluating the precision in this case. If estimated SSB_{2007}/SSB_{msy} is a minimum at $M = 0.2\,yr^{-1}$, and is higher at values of M both immediately above and below $0.2\,yr^{-1}$, it can be concluded that the expected value of this ratio is higher than is reported in the assessment over the range of possible M. In fairness, assessments (whether VPA or maximum likelihood) usually report modal values, and cannot report expected values without further analysis such as a Monte Carlo exploration. An approximation of the expected value can be obtained by another form of the delta method, which uses second partial derivatives to estimate bias (viz. Seber, 1973), but that application is beyond the scope of this paper.

4. Discussion

The delta method provides a practical solution to the problem of estimating precision of assessment quantities when some parameters are pre-specified. The calculations are easy to implement in a spreadsheet format and can be conducted in a few hours for most VPA or SS assessments. Delta method results tend to be comparable to asymptotic precision estimates from SS if the parameters are estimated. Thus, to the extent that asymptotic estimates of precision are routinely presented in stock assessments, a more complete accounting of uncertainty by means of the delta method has analogous and possibly equivalent properties. Although Bayesian posterior distributions of assessment quantities are more complete, they are not possible for VPAs and often require many days of computing for SS in the rare cases where they are derived at all.

The reader must be cautioned to avoid generalizing patterns of imprecision from the two examples provided here. Data properties and model specifications can influence imprecision in ways that are difficult to anticipate. For example, length-based assessments can be especially sensitive to the value of σ_R. Fortunately, the calculations needed for the delta method are easily conducted in a short time, and there is no obstacle to fully evaluating the approximate precision of most assessment models.

Acknowledgments

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Appendix A. A spreadsheet implementation of the delta method

Each row in the following spreadsheet produces a delta method estimate of the precision of a corresponding quantity reported in the *.std list produced by Stock Synthesis or ADMB.

Columns are indicated by letters:

Section 1: Base model with fixed parameters

A. Names of estimated quantities
(column of names is pasted from element 2 of the *.std file of base model)
B. Estimated values with fixed parameters
(column of estimates from element 3 *.std file of base model)
C. Estimated standard deviations
(column of base standard errors from element 4 *.std file of base model)
Note, columns A–C are cut from *.std and pasted in a single operation.

Section 2: Numerical estimation of partial derivatives

Select appropriate size of difference, Δ_M.
(Caution, precision of M may elsewhere be expressed as $\sigma_{\ln(M)}$, causing confusion.)
D. Estimated values of quantities using fixed parameter $M+\Delta_M$
(estimates are pasted from element 3 of *.std file from output of model with $M+\Delta_M$)
E. Estimated values using fixed parameter $M-\Delta_M$ (pasted from *.std file)
Inspect values to determine if a smaller Δ_M is needed.
F. Partial derivative with respect to M: column F = (column D − column E)/2Δ_M
Select appropriate size of finite difference, Δ_h
G. Estimated values using fixed parameter $h+\Delta_h$ (pasted from *.std file)
H. Estimated values using fixed parameter $h-\Delta_h$ (pasted from *.std file)
Inspect values to determine if a smaller Δ_h is needed.
I. Partial derivative with respect to h: column I = (column G − column H)/2Δ_h
Repeat sets of three columns for additional fixed parameters, σ_R, etc.

Section 3: Estimation of variance components, using σ_M (not $\sigma_{\ln(M)}$), σ_h, and possibly cov(M, h)

J. Base model variance is column C × column C
K. Variance component from M is column F × column F × $\sigma_M \times \sigma_M$
L. Variance component from h is column I × column I × $\sigma_h \times \sigma_h$
M. If covariance (M, h) is non-zero, covariance component is 2 × column F × column I × cov(M, h)

Section 4: Delta method calculation

N. Estimated total variance is column J + column K + column L + column M
O. Estimated standard error is SQRT (column N)
P. Estimated coefficient of variation is column B/column
Q. Fraction of variance from base is column J/column N
R. Fraction of variance from M is column K/column N
S. Fraction of variance from h is column L/column N
T. Relative size of covariance correction is column M/column N

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